The Computation of Coaxial Line Step Capacitances

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Abstract—**Accurate values of coaxial line step capacitances have been computed by the method of 'mode matching' at the plane of the discontinuity in a coaxial line, as described by Whinnery, Jamieson, and Robbins in 1944. The capacitances are given in tables, graphs, and as empirical explicit expressions, suitable for programming in digital computers.**

Introduction

STEP CAPACITANCE, or discontinuity capacitance, is the name given to the capacitance which occurs in transmission lines at the plane of a sudden change of geometry (discontinuity), and causes reflections, apart from reflections caused by any sudden change of the characteristic impedance of the line. As an example, if the dimensions of a coaxial line from a given transverse plane onwards are suddenly scaled up, thus keeping constant the characteristic impedance of the transmission line, this line will no longer be reflectionless because of the disturbance of the field pattern in the vicinity of the discontinuity.

Whinnery and Jamieson [1] have described the detailed physical picture of the phenomena taking place at the 'step,' and have justified the concept of an equivalent shunt capacitance located that the plane of the discontinuity. In a subsequent article by the same authors [2], mathematical methods were presented for the evaluation of this capacitance from the dimensions, the frequency and the properties of the dielectric present in a coaxial line. The results were given in the form of charts (Figs. 8 and 9 in reference [2]) which up to the present time have been the only theoretical source of numerical information on the subject, apart from electrostatic methods, which, of course, do not take into account the effect of frequency.

In an earlier communication [3] an explicit formula was presented which enabled values of step capacitances from Whinnery, Jamieson, and Robbins's charts [2] to be used in digital computer programs.

Later it was shown]4], [5] that this formula had reproduced inaccuracies inherent in the original charts. Subsequent investigation has shown that the charts are up to 5 percent in error.

Precision measurements have now progressed to a stage where more accurate charts are needed. As an illustrative example, the value of shunt capacitance in a 50 ohm coaxial line operated at 1 GHz, that would cause a VSWR of 1.001, is of the order of 3 fF (3×10^{-15} F). This means that if the measurement is capable of resolving a VSWR of 1.001, any step capacitances should be known to an accuracy of approximately ± 0.1 fF to ensure that errors introduced by inaccurate values of step capacitances do not limit the interpretation of the measurement.

Such a requirement, and the realization that methods like relaxation, or the Monte Carlo method would be too lengthy, even on a high speed computer, to produce an answer of five accurate digits, which even then would apply only at zero frequency, led to the programming of the method described by Whinnery, Jamieson and, Robbins [2] for a high speed digital computer.

In the next section the computed results are presented in three ways: in graphical form; tabulated; and in a form suitable for inclusion in digital computer programs.

The Computed Values of Step Capacitance

Graphical Form

For the benefit of users wishing to obtain values of step capacitance quickly, where an accuracy of about one percent is sufficient, the results have been plotted (Figs. 1 and 2). Because, at small values of α,[1] the step capacitance in a plane line varies quasi-logarithmically. a semi-logarithmic chart was used. (Using the identity $\cosh^{-1}x=\ln(x+\sqrt{x^2-1})$ in the explicit expression for the step capacitance in a plane line [1] we obtain

$$C_d=\frac{\epsilon}{\pi}\left[\frac{\alpha^2+1}{\alpha}\ln\frac{1+\alpha}{1-\alpha}-2\ln\frac{4\alpha}{1-\alpha^2}\right]F/m$$

which reveals that

$$C_d\approx\frac{2\epsilon}{\pi}[1-\ln 4\alpha],$$

for small values of α.)

The computed values of step capacitance at zero frequency given in Table I are believed to be accurate to about five digits. They were obtained by taking into account the first 40 higher modes of propagation, and extrapolating hyperbolically the returned values of step capacitances as a function of numbers of modes used to produce them, and regarding the asymptotic value of this hyperbola as the final answer. (For further details of this extrapolation see the discussion on accuracy.)

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[1] For the definition of α, see Figs. 1 and 2.

Inclusion in Computer Programs

In addition to the full-scale computer program described in this paper, there are several other methods, faster but of limited accuracy, whereby step capacitance values may be obtained using a digital computer.

One approach would be to store the numerical values given in Table I in the machine, and interpolate between points in the required region. As an alternative, one may produce approximate (but sufficiently accurate) explicit expressions which, within set limits of the arguments, will return the values of step capacitance. One such possibility was suggested by the author [3]. Another possibility is to take advantage of the fact that the effect of curvature on the step capacitance is small, therefore all values of step capacitances for $\tau \neq 1$ may be regarded as a perturbation on the values for $\tau = 1$ (the plane case). The result of this approach is given in the following expressions.

Step on inner

$$C_d = \frac{\epsilon}{100\pi}\left[\frac{\alpha^2+1}{\alpha}\ln\frac{1+\alpha}{1-\alpha} - 2\ln\frac{4\alpha}{1-\alpha^2}\right] + 1.11 \times 10^{-15}(1-\alpha)(\tau-1)\mathrm{F/cm}.$$

This formula produces a maximum error of ± 0.3 fF/cm for the ranges $0.01 \leq \alpha < 1.0$ and $1.0 \leq \tau \leq 6.0$.

Step on outer

$$C_d = \frac{\epsilon}{100\pi}\left[\frac{\alpha^2+1}{\alpha}\ln\frac{1+\alpha}{1-\alpha} - 2\ln\frac{4\alpha}{1-\alpha^2}\right] + 4.12 \times 10^{-15}(0.8-\alpha)(\tau-1.4)\mathrm{F/cm}.$$

The maximum error produced by this formula is ± 0.6 fF/cm for the ranges $0.01 \leq \alpha \leq 0.7$ and $1.5 \leq \tau \leq 6.0$.

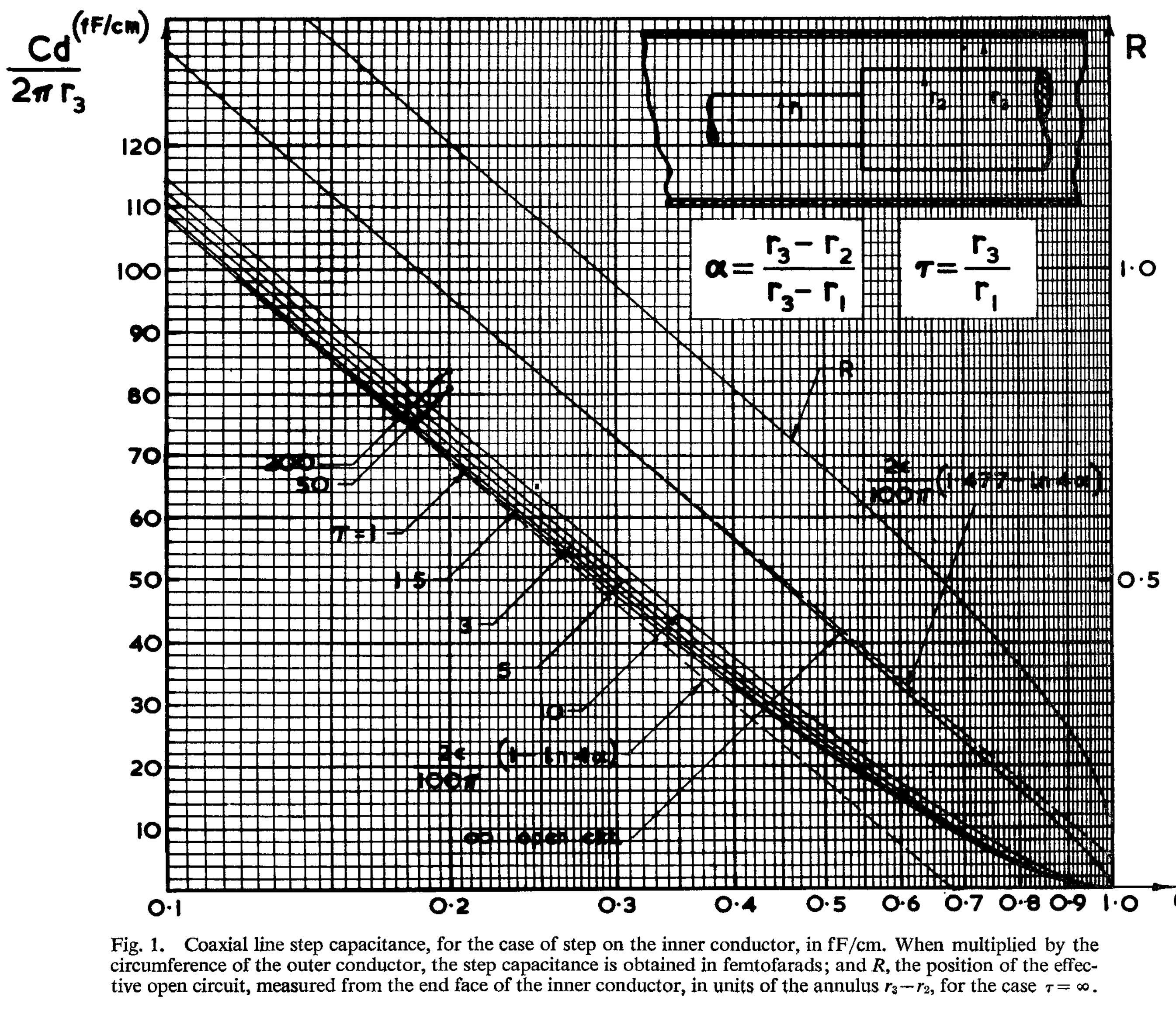

Fig. 1. Coaxial line step capacitance, for the case of step on the inner conductor, in fF/cm. When multiplied by the circumference of the outer conductor, the step capacitance is obtained in femtofarads; and R, the position of the effective open circuit, measured from the end face of the inner conductor, in units of the annulus $r_3 - r_2$, for the case $\tau = \infty$.

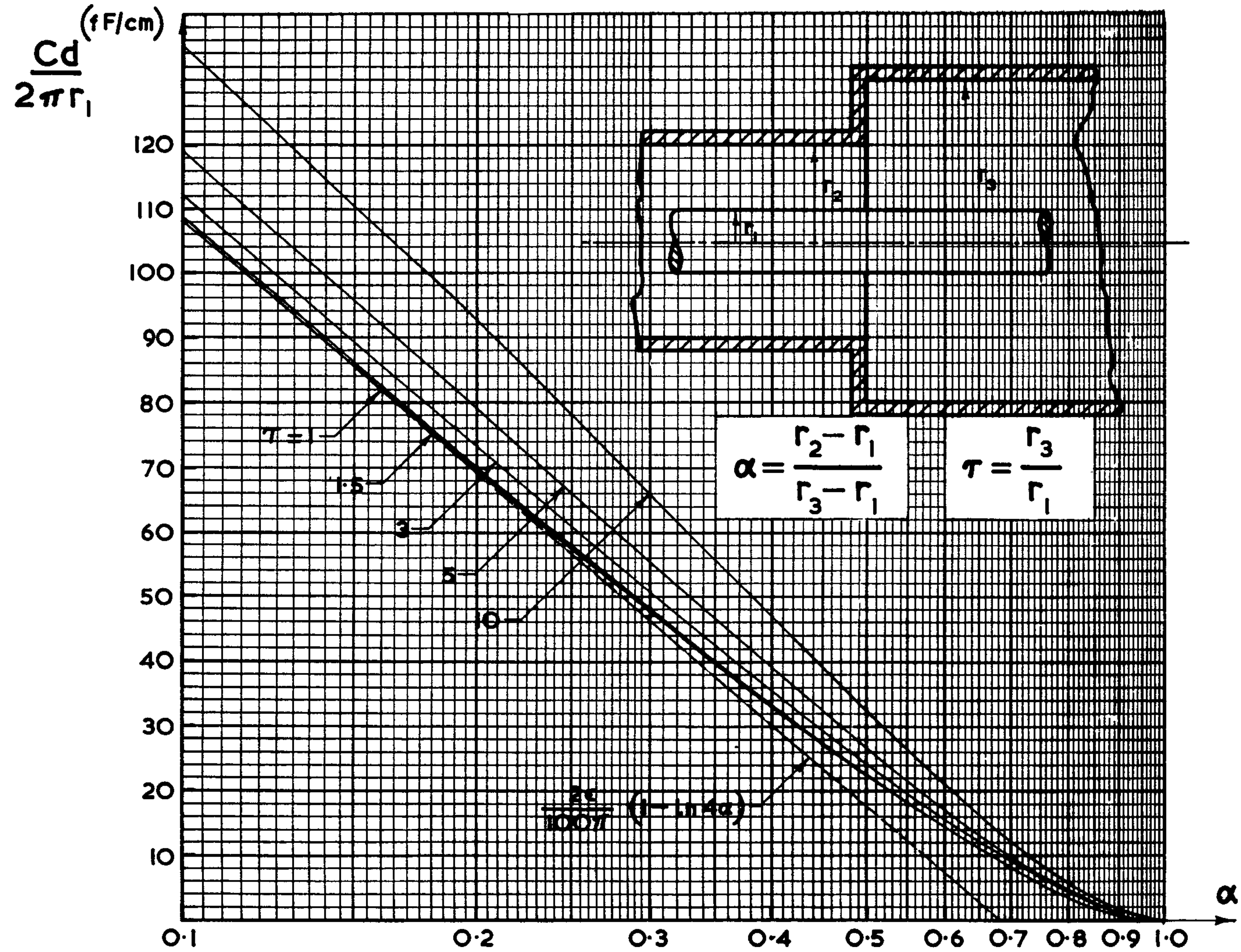

Fig. 2. Coaxial line step capacitance, for the case of step on the outer conductor, in fF/cm. When multiplied by the circumference of the inner conductor, the step capacitance is obtained in femtofarads.

TABLE I

Computed Values of Step Capacitance in fF/cm

When multiplied by the circumference of the unstepped conductor, these figures yield the value of step capacitance in femtofarads

α	Plane Case	Step on Inner				Open Circuit	Step on Outer			
	$\tau=1$	1.5	3	5	10	∞	1.5	3	5	10
0.1	108.20	108.56	110.31	112.05	114.56	134.89	108.66	112.42	118.82	135.07
0.2	69.700	70.024	71.663	73.320	75.736	95.659	70.111	73.398	78.814	92.659
0.3	47.797	48.083	49.562	51.089	53.349	72.607	48.149	50.885	55.265	65.767
0.4	32.931	33.171	34.452	35.808	37.859	56.171	33.218	35.388	38.774	46.741
0.5	22.121	22.311	23.363	24.514	26.302	43.358	22.341	23.965	26.445	32.194
0.6	14.057	14.196	14.999	15.913	17.387	32.832	14.213	15.338	17.025	20.887
0.7	8.0688	8.1592	8.7063	9.3604	10.469	23.869	8.1665	8.8597	9.8814	12.199
0.8	3.7965	3.8436	4.1462	4.5311	5.2314	16.009	3.8455	4.1909	4.6926	5.8217
0.9	1.0784	1.0926	1.1927	1.3303	1.6095	8.8579	1.0927	1.1952	1.3423	1.6712
1.0	0.0	0.0	0.0	0.00	0.0	0.0	0.0	0.0	0.0	0.0

Accuracy of Results

There are three possible sources of error in the calculation of the step capacitance. These are:

1) the limited number of digits representing a floating point quantity in the computer;
2) the threshold values set in the program to terminate iterative processes, etc.;
3) the extrapolation used in obtaining the final values of step capacitances, i.e. using only a finite number of higher order modes to predict the asymptotic values of the step capacitances corresponding to an infinite number of modes, as required by the theory.

The computer employed had a word length of 48 bits, resulting approximately in 12 decimal digits. The effect on accuracy of a finite number of digits is most serious when differences of similar quantities are taken, in which case several significant digits are lost, despite the fact that the computer still produces an answer containing 12 decimal digits. In critical places in the program where this was likely to happen double precision was used, and it is believed that errors from this source do not contribute to the first five digits of the results.

For the set threshold values of iterative processes the check is more positive. It is always possible to change the value of the threshold and observe the effect of this change on the final answer. The value of the threshold should be such that further reduction does not affect the result significantly.

As mentioned before, the theory [2] requires an infinite number of higher order modes of propagation to be taken into account. This is not possible. In the original calculation by Whinnery, Jamieson, and Robbins, made before automatic digital computers became available, only the first four modes were taken into account. In the present calculation provision was made to use the first 140 modes, but it was found that the required accuracy could be achieved by using only 40 modes and extrapolating in the following way to obtain an answer closely approaching that corresponding to an infinite number of modes.

In analyzing one point ($\alpha=0.5$, $\tau=3.0$), results were computed using the first 28, 29, $\cdots$ 40 modes. It was found that the computed step capacitances considered as a function of the numbers of modes used could be approximated very well by an hyperbola of the form $C=A/m+B$, where A and B are constants. The coefficient of correlation between m and C for the above-mentioned case was found to be 0.999 992. Thus it is believed that the asymptotic value B of the first order hyperbola fitted by the method of least squares may be regarded as the solution within the limits of accuracy given above.

As a further check, using the above procedure, for a typical value of $\alpha=0.5$, a set of capacitances was calculated for $\tau=1.5$, 1.4, 1.3, 1.2, and 1.1. Extrapolating this set to $\tau=1.0$, which is that of the plane case for which an explicit formula exists (given above), the residual error was found to be in the order of 0.01 percent.

Any further increase in computational accuracy is hardly warranted in most practical applications because of the limited accuracy of the knowledge of the physical dimensions the departure of the actual geometry from the theoretical (perfectly smooth concentric circular cylinders having perfectly sharp corners), and conductor losses.

The Effect of Frequency

As has been pointed out [1], [2], it is not possible to present universally valid frequency correction factors K,[2] which when multiplied by the step capacitance values for zero frequency, yield the step capacitances at the desired frequencies because each different case has a different frequency correction factor. K is a function of α, τ, and the frequency, and also depends upon whether the step is on the inner or the outer conductor.

However, to indicate generally the form and magnitude of the frequency correction factor, K is given in Fig. 3.

For users wishing to obtain values of step capacitances accurate to five digits at non-zero frequencies, the only possible way seems to be to run the complete program on a computer for the case in question. To those who are interested, the author would be pleased to send a copy of the computer program listing written in CDC 3600 FORTRAN.

The Method of Programming

The computations of 'step on inner' and 'step on outer' are very similar, therefore only the case of the 'step on inner' is described. The equation numbers, unless stated otherwise, refer to reference [2].

The program starts with the evaluation of quantities derived from Bessel functions. For arguments <50 the 'backward recurrence' method is used to evaluate the Bessel functions [6] and the roots of the transcendental equations (7) and (8) are found by iteration. For arguments≤ 50 McMahon's method [7] is used to yield the roots. From the desired number of roots, the quantities k_{Am}, k_{Bn}, G_{Am}, and G_{Bn} of (5) and (6) and finally the values of Z of (3) are obtained. The given frequency determines K_{Am} and K_{Bn} of (16).

The above quantities, and the Hahn functions (80), (81), and (82) of reference [1], yield the quantities L_o, L_p, M_p of (33), (34), and (35), and thus the elements of the matrix equation (37) may be formulated. The solutions of the matrix equation are used in (36) to work out the step capacitance corresponding to the given number of higher order

[2] In this paper the symbol for the frequency correction factor was chosen to be K in contrast with reference [2] where F was used, to avoid confusion now that F has become the accepted symbol for Farads.

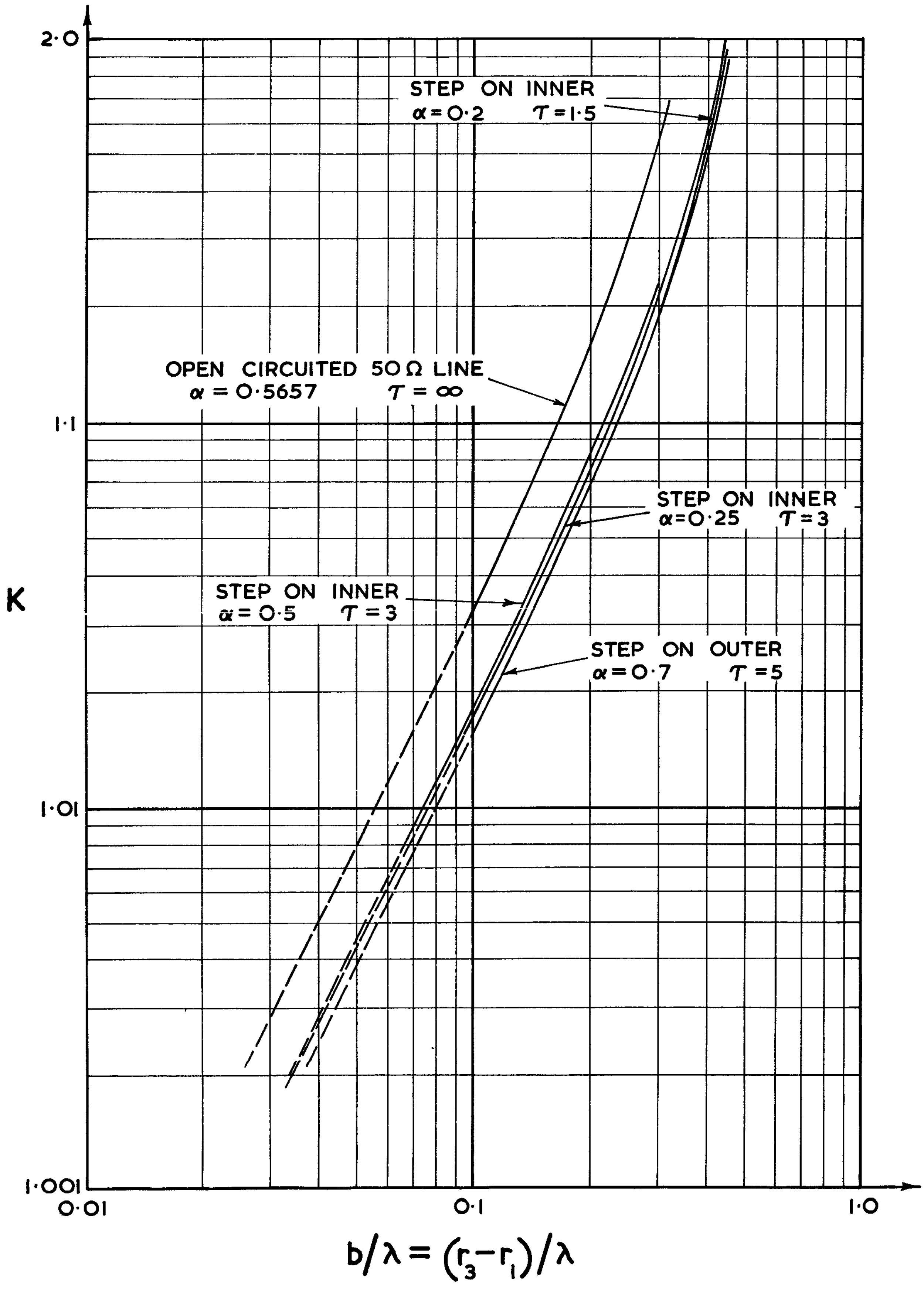

Fig. 3. Frequency correction factor for coaxial line step capacitance vs. $(r_3-r_1)/\lambda$.

modes. Next, the number of modes is increased by one, and a different value of capacitance is obtained. Then the number of modes is increased again by one, and this is repeated until the highest desired number of evanescent modes have been used. Finally, a first order hyperbola is fitted by the method of least squares, to the capacitances as functions of number of modes used. The asymptotic value of this hyperbola is regarded as the final value of capacitance. It has been found that using the first 28, 29, . . . , 40 modes yields a sufficiently accurate asymptotic value for the 'best fit' hyperbola.

It should be noted that a few misprints were located in reference [2] which had to be corrected before the method could be used. In (37) and (45) the subscripts m and p should be reversed to comply with the accepted order of subscripts of matrix elements, where the first subscript stands for row number and the second for column number. In (44), in the denominator of the second term the quantity in brackets should not be squared and the subscript of r should be 1 instead of 3. In the same equation and in (45) the arguments of the L and M functions should not be $(b-a)/b$ as indicated, but only a/b, once the appropriate definitions of a and b have been set down for the condition 'step on outer'—as given in the text. Similarly, in the numerator of the first term of (44), the quantity indicated as $(b-a)^2$ should simply be a^2. Finally, in the numerator of the last summation of (32), $(k_{Bn}\sqrt{2})^3$ should read $(k_{Bn}r_2)^3$.

Conclusions

A long-standing need to obtain more accurate values of coaxial line step capacitances has been satisfied, using the approach suggested by Hahn in 1941, and applied in detail by Whinnery, Jamieson, and Robbins in 1944, by programming the above-mentioned method for a high speed digital computer.

Acknowledgment

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